# Spherical Harmonics-based Visibility Synthesis for Cosmological Intensity Mapping

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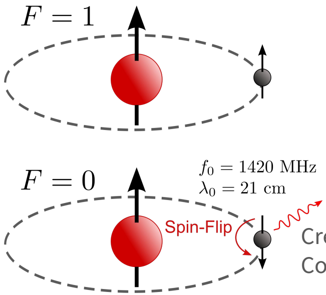
# HI Intensity Mapping Tomography

- Hyperfine Hydrogen transition line at 1420.4 MHz
- Efficiently and tomographically map cosmological volumes
  - Generally low angular resolution but redshift information cheap
  - Probe epoch of reionisation at low frequencies and large scale structure at high frequencies.




Credit: Wikimedia Commons

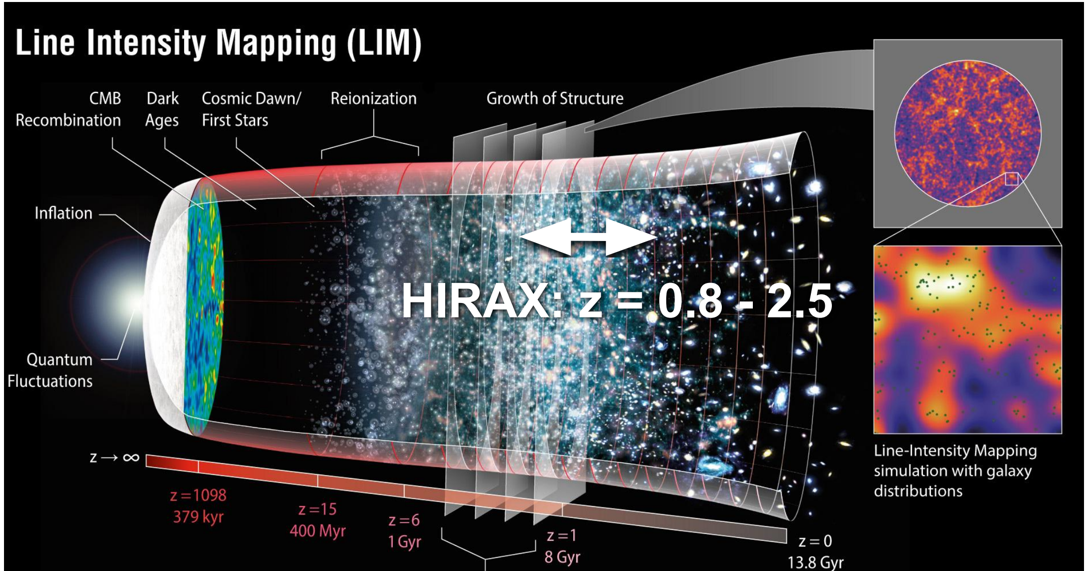
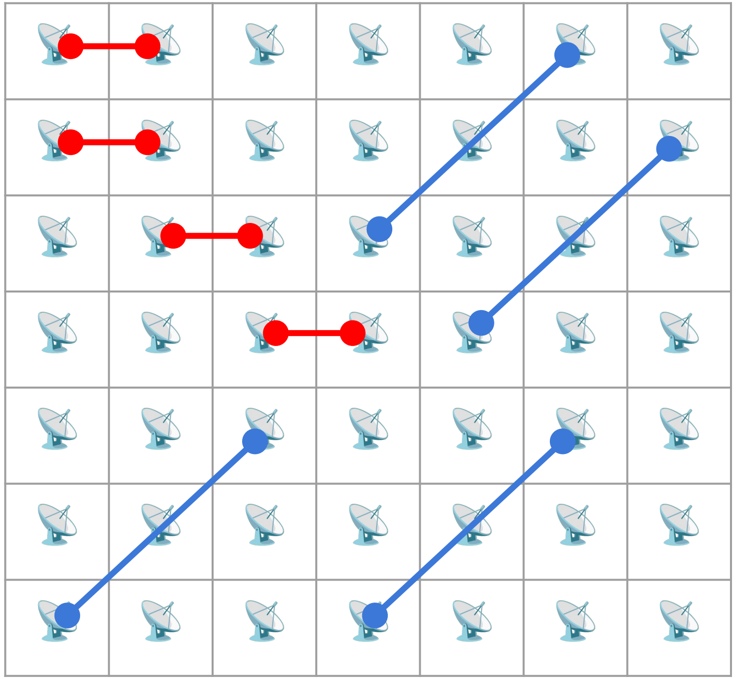



Image Credit: NASA / LAMBDA Science Team

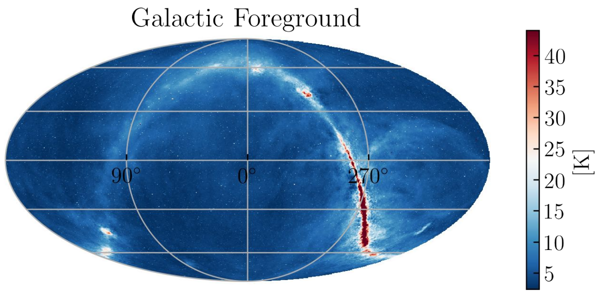
# Motivation for Compact Redundant Arrays

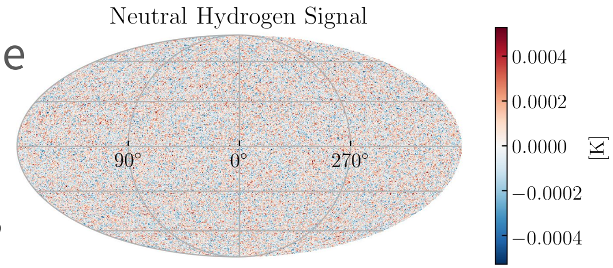
- Compact - weight on short baselines
  - Targeting large angular scales
  - Potential for cross-talk, reflections and impact from array-level effects
- Redundant - Large N with repeated baselines
  - Enhanced sensitivity to sky Fourier modes on interest
  - Internal, redundant calibration
    - If instrument redundancy is realised
  - On-site calibration a requirement for keeping up with data transfer off-site
- E.g HIRAX, CHIME, CHORD, HERA, MWA



# Systematics / Chromaticity and Foregrounds

- Foregrounds are the primary challenge for 21cm cosmology
  - Galactic signal brighter by many orders of magnitude
- Signal and Foregrounds have different, *on-sky* properties
  - Galactic emission is:
    - Polarised
    - Strongly correlated over wide frequency bands
    - Structured on the sky in ~known way
  - In principle, there are not many mixed *on-sky* degrees of freedom
- Mode-mixing inherent in measurement/cal. is a major issue
  - Instrument has chromatic response *fundamentally* as well as arising from *systematics*
  - With perfect knowledge of the instrument, this can be accounted for, however the large contrast in signal strengths can make small reconstruction residuals a big problem




**Motivation for non-redundant instrument simulation for large -N**

# Spherical Harmonic Visibility Synthesis

Primary Beams　　Baseline Term　　Sky Signal

$$\mathcal{V}^{ij\nu}(\phi) = \int d\Omega\ A^{i\nu} A^{j\nu *}(\hat{\boldsymbol{\theta}})\ \exp\left(-2\pi i\ \boldsymbol{u}^{i-j} \cdot \hat{\boldsymbol{\theta}}\right) T^{\nu}(\hat{\boldsymbol{\theta}}, \phi)$$

We want to evaluate this integral to forward model for many distinct *i* and *j*

Why Spherical harmonics?

- Appropriate for wide field instruments (horizon-to-horizon simulations)
- M-mode separability (see later) / azimuthal sky rotation for **driftscan**
- Analytic tools (e.g. spherical harmonic transform of baseline term)
- Can be compact representation of e.g. beam terms (~azimuthal symmetry)

# Standard m-mode (fixed telescope, flexible sky)

Visibility as function of earth rotation angle ϕ

$$\mathcal{V}^{ij\nu}(\phi) = \int d\Omega \left[A^{i\nu} A^{j\nu*}(\hat{\boldsymbol{\theta}}) \ \exp\left(-2\pi i \ \boldsymbol{u}^{i-j} \cdot \hat{\boldsymbol{\theta}}\right)\right] \left[T^{\nu}(\hat{\boldsymbol{\theta}}, \phi)\right]$$

SHT [beams x baseline] and sky

$$\mathcal{V}^{ij\nu}(\phi) = \int d\Omega \left[\sum_{\ell_B m_B} B^{ij\nu}_{\ell_B m_B} \ Y^{*}_{\ell_B m_B}(\hat{\boldsymbol{\theta}}_{\mathrm{TIRS}})\right] \left[\sum_{\ell_a m_a} a^{\nu}_{\ell_a m_a} Y_{\ell_a m_a}(\hat{\boldsymbol{\theta}}_{\mathrm{CIRS}})\right]$$

$$= \sum_{\ell_B m_B \ell_a m_a} B^{ij\nu}_{\ell_B m_B} a^{\nu}_{\ell_a m_a} \int d\Omega \ Y^{*}_{\ell_B m_B}(\hat{\boldsymbol{\theta}}_{\mathrm{TIRS}}) Y_{\ell_a m_a}(\hat{\boldsymbol{\theta}}_{\mathrm{CIRS}})$$

Azimuthal basis rotation of sky SH's CIRS→TIRS

$$= \sum_{\ell_B m_B \ell_a m_a} B^{ij\nu}_{\ell_B m_B} a^{\nu}_{\ell_a m_a} \int d\Omega \ Y^{*}_{\ell_B m_B}(\hat{\boldsymbol{\theta}}_{\mathrm{TIRS}}) Y_{\ell_a m_a}(\hat{\boldsymbol{\theta}}_{\mathrm{TIRS}}) e^{i m_a \phi}$$

Integral of 2 SH's gives delta functions

$$= \sum_{\ell_B m_B \ell_a m_a} B^{ij\nu}_{\ell_B m_B} a^{\nu}_{\ell_a m_a} \boxed{\delta_{\ell_B \ell_a} \delta_{m_B m_a}} e^{i m_a \phi}$$

$$= \sum_{\ell m} B^{ij\nu}_{\ell m} a^{\nu}_{\ell m} e^{i m \phi}$$

**TIRS**
Fixed to earth
**CIRS**
Fixed to Sky

Related by pure azimuthal sidereal rotation

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Developed in Shaw et al. 2024 and 2015

# Standard m-mode (fixed telescope, flexible sky)

$$\mathcal{V}^{ij\nu}(\phi) = \int d\Omega \left[A^{i\nu} A^{j\nu *}(\hat{\boldsymbol{\theta}}) \ \exp\left(-2\pi i \ \boldsymbol{u}^{i-j} \cdot \hat{\boldsymbol{\theta}}\right)\right] \left[T^{\nu}(\hat{\boldsymbol{\theta}}, \phi)\right]$$

$$= \sum_{\ell m} B^{ij\nu}_{\ell m} a^{\nu}_{\ell m} e^{im\phi}$$

Fourier transform with respect to earth rotation angle ϕ

$$\mathcal{V}^{ij\nu}_{m'} = \int \frac{\mathrm{d}\phi}{2\pi} \sum_{\ell m} B^{ij\nu}_{\ell m} a^{\nu}_{\ell m} e^{i(m-m')\phi}$$

$$= \sum_{\ell m} B^{ij\nu}_{\ell m} a^{\nu}_{\ell m} \delta_{mm'}$$

Compact linear map for m-mode visibilities

$$\mathcal{V}^{ij\nu}_{m} = \sum_{\ell} B^{ij\nu}_{\ell m} a^{\nu}_{\ell m}$$

**M-Modes**

Each Fourier component of visibility with earth rotation angle (~time) picks out a specific sky ***m.*** This is very powerful.

Get full sidereal day of data for ~same cost of naively computing one time step

Have efficient linear projection from sky to visibilities for full sidereal days.

- ***But*** to compute the beam transfer matrix, telescope model is fixed in cache
- Flexible telescope (e.g. primary beams) is not efficiently computable



Developed in Shaw et al. 2024 and 2015

# Visibility synthesis for driftscan telescopes

M-mode approach for systematics simulation:

- With known instrument, can efficiently predict data as sky varies
  - But hard to vary instrument without recomputing many harmonic transforms
- Very efficient and m-separability very useful, particularly for inverse problems
- Instrument model baked in to beam transfer matrices and they must be recomputed if it’s updated, not ideal for systematics
- Requires primary beams evaluated on same grid as the baseline phase term and sky
  - Can’t make use of band limits of individual terms separately.
- Can we does something similar with primary beams flexible?

# Triple SH m-mode (as flexible as you want)

$$\mathcal{V}^{ij\nu}(\phi) = \int d\Omega \left[A^{i\nu} A^{j\nu *}(\hat{\boldsymbol{\theta}})\right] \left[\exp\left(-2\pi i \, \boldsymbol{u}^{i-j} \cdot \hat{\boldsymbol{\theta}}\right)\right] \left[T^{\nu}(\hat{\boldsymbol{\theta}}, \phi)\right]$$

SHT beams, baselines and sky

$$\mathcal{V}^{ij\nu}(\phi) = \int d\Omega \left[\sum_{\ell_p m_p} p^{ij\nu}_{\ell_p m_p} \, Y_{\ell_p m_p}(\hat{\boldsymbol{\theta}}_{\rm TIRS})\right] \left[\sum_{\ell_u m_u} u^{i-j,\nu}_{\ell_u m_u} \, Y_{\ell_u m_u}(\hat{\boldsymbol{\theta}}_{\rm TIRS})\right] \left[\sum_{\ell_a m_a} a^{\nu}_{\ell_a m_a} Y_{\ell_a m_a}(\hat{\boldsymbol{\theta}}_{\rm CIRS})\right]$$

Azimuthal basis rotation of sky SH's CIRS→TIRS

$$= \sum_{\ell_{pua}, m_{pua}} p^{ij\nu}_{\ell_p m_p} u^{i-j,\nu}_{\ell_u m_u} a^{\nu}_{\ell_a m_a} \int d\Omega \, Y_{\ell_p m_p}(\hat{\boldsymbol{\theta}}_{\rm TIRS}) Y_{\ell_u m_u}(\hat{\boldsymbol{\theta}}_{\rm TIRS}) Y_{\ell_a m_a}(\hat{\boldsymbol{\theta}}_{\rm TIRS}) e^{i m_a \phi}$$

Integral of 3 SHs gives Gaunt coeff.

$$= \sum_{\ell_{pua}, m_{pua}} p^{ij\nu}_{\ell_p m_p} u^{i-j,\nu}_{\ell_u m_u} a^{\nu}_{\ell_a m_a} e^{i m_a \phi} \mathcal{G}^{\ell_p \;\ell_u \;\ell_a}_{m_p \; m_u \; m_a}$$

Triple spherical harmonic integral maintains m-mode separability but now we can sum over whatever term(s) we want to fix and keep others flexible

- But need Gaunt coefficients… not as nice as delta functions but not so bad.

# Triple SH m-mode (fix sky, flexible beams)

$$\mathcal{V}^{ij\nu}(\phi) = \int d\Omega \left[A^{i\nu}A^{j\nu*}(\hat{\boldsymbol{\theta}})\right] \left[\exp\left(-2\pi i\, \boldsymbol{u}^{i-j}\cdot\hat{\boldsymbol{\theta}}\right)\right] \left[T^{\nu}(\hat{\boldsymbol{\theta}},\phi)\right]$$

$$= \sum_{\ell_{pua},m_{pua}} p^{ij\nu}_{\ell_p m_p} u^{i-j,\nu}_{\ell_u m_u} a^{\nu}_{\ell_a m_a} e^{i m_a \phi} \mathcal{G}^{\ell_p\ \ \ell_u\ \ \ell_a}_{m_p\ m_u\ m_a}$$

Fourier transform with respect to earth rotation angle ϕ

$$\mathcal{V}^{ij\nu}_m = \sum_{\ell_{pua},m_{pu}} p^{ij\nu}_{\ell_p m_p} u^{i-j,\nu}_{\ell_u m_u} a^{\nu}_{\ell_a m} \mathcal{G}^{\ell_p\ \ \ell_u\ \ \ell_a}_{m_p\ m_u\ m}$$

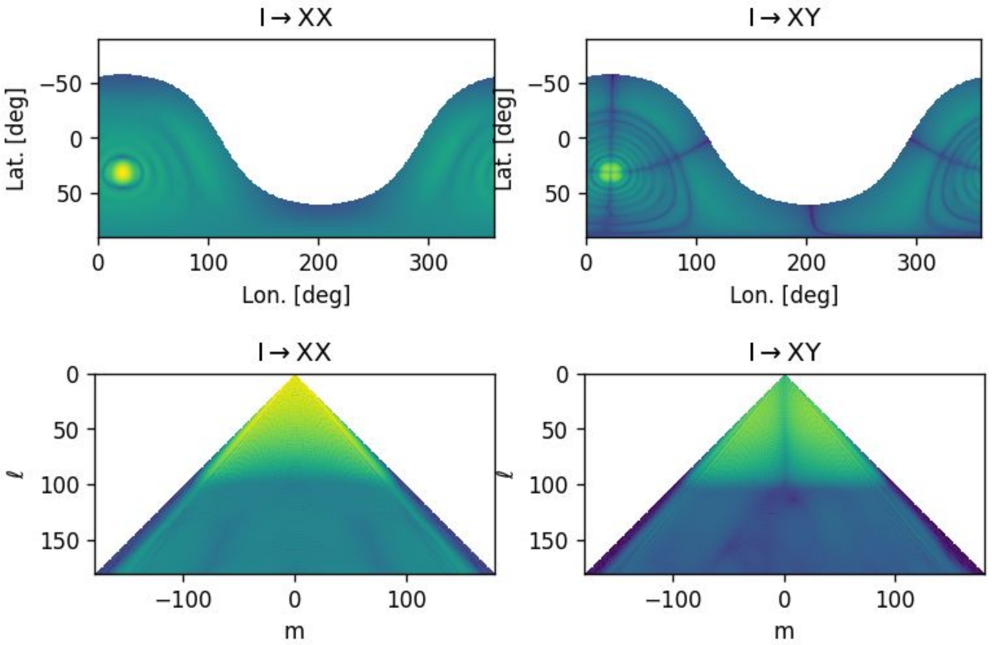
Sum over baseline harmonics (u) and sky harmonics (a)

$$\boxed{\mathcal{V}^{ij\nu}_m = \sum_{\ell_p,m_p} P^{i-j,\nu}_{l_p m_p m} p^{ij,\nu}_{\ell_p m_p}}$$

Here we do what we typically want for systematic non-redundancy studies. Fix the sky and choose some baselines. Then we have an m-separable linear model for the visibilities as a function of power primary beam spherical harmonics.

- Power primary beam harmonics can be generated from voltage beams with similar methods

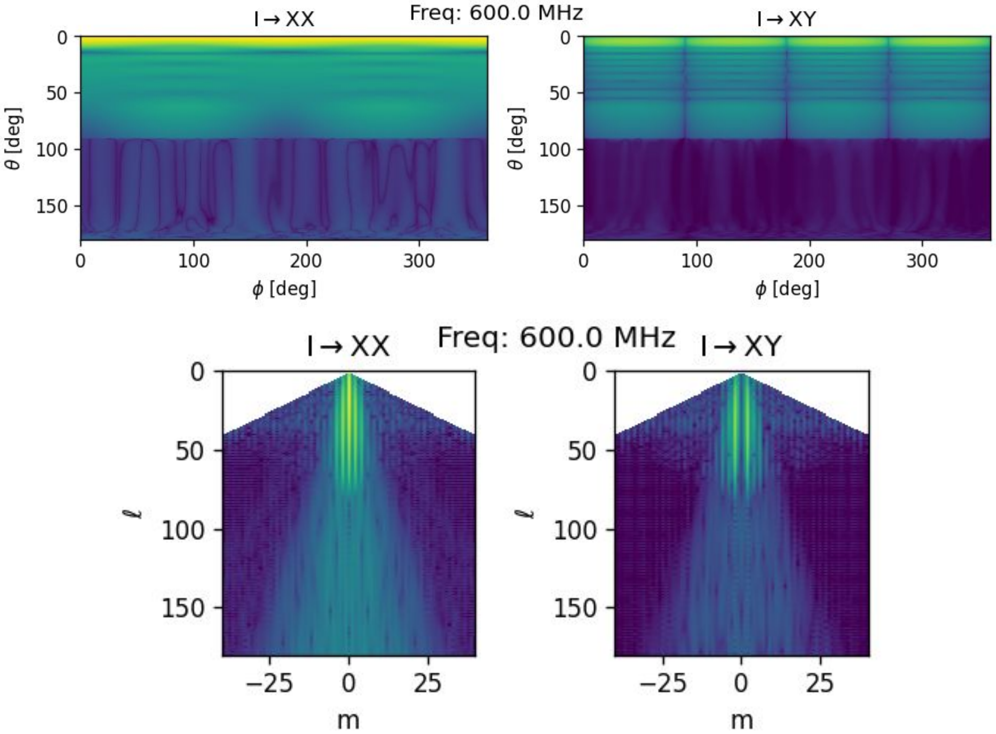
# Aside: Beam m-compactness




HIRAX beams are compact in ℓ but, in TIRS, they are not compact in m.

However, they are quite azimuthally symmetric around the pointing direction, even e.g. XY terms

# Aside: Beam m-compactness




But we can represent them in a pointing-oriented coordinate system.

Then they are compact in ℓ and m. We just need to include a SH rotation term in our cache. (Wigner-D matrices).

If we do so, the cache and beam representations can be extremely compact

# SERVAL

## Spherical Expansions for Radio Visibility Analysis at Large-N

CODE | DOCS - Authors: myself and Marianna Papadionysiou

Python Package with C++ backend

- Computes Gaunt coefficients
- Efficiently sums over them in the ways we want for systematics foreward modelling
- Allows Gaunt-based generation of beam transfer matrix or fixed-sky visibility projector
- Misc tools for spherical harmonics and rotations

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# SERVAL

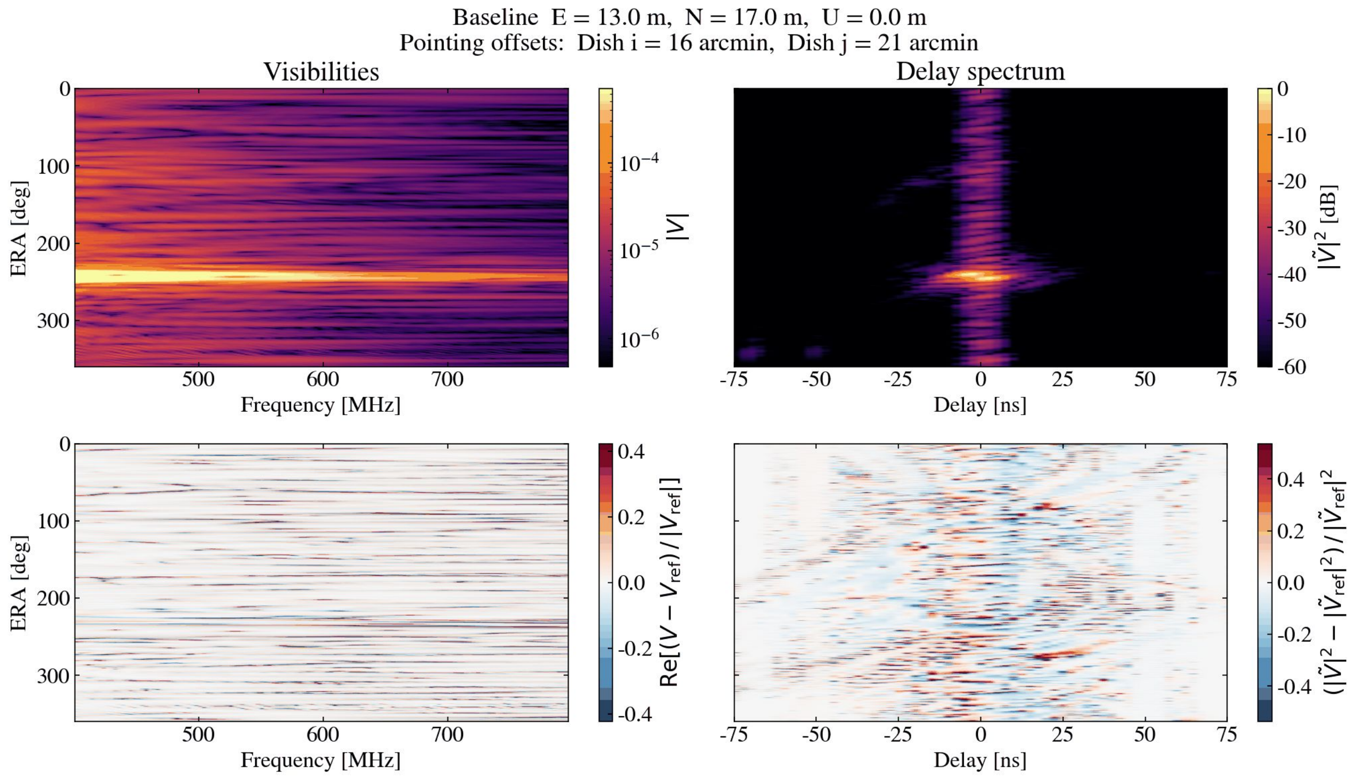
## Spherical Expansions for Radio Visibility Analysis at Large-N

Status: **Early release available. Final release and publication coming soon**

- Mostly feature complete with significant test coverage
    - Includes tests with known analytic results
- Benchmarking and performance improvements ongoing but significant progress has been made in the last months. Performance is very reasonable for typical HIRAX-scale problems
- Documentation is substantial but incomplete and being worked on
- Happy to discuss usage with anyone interested!

# Example Results - Pointing Offset Airy Beams

Visibilities for a library of voltage beams from pointing-perturbed dishes



# Example Results - Pointing Offset Airy Beams

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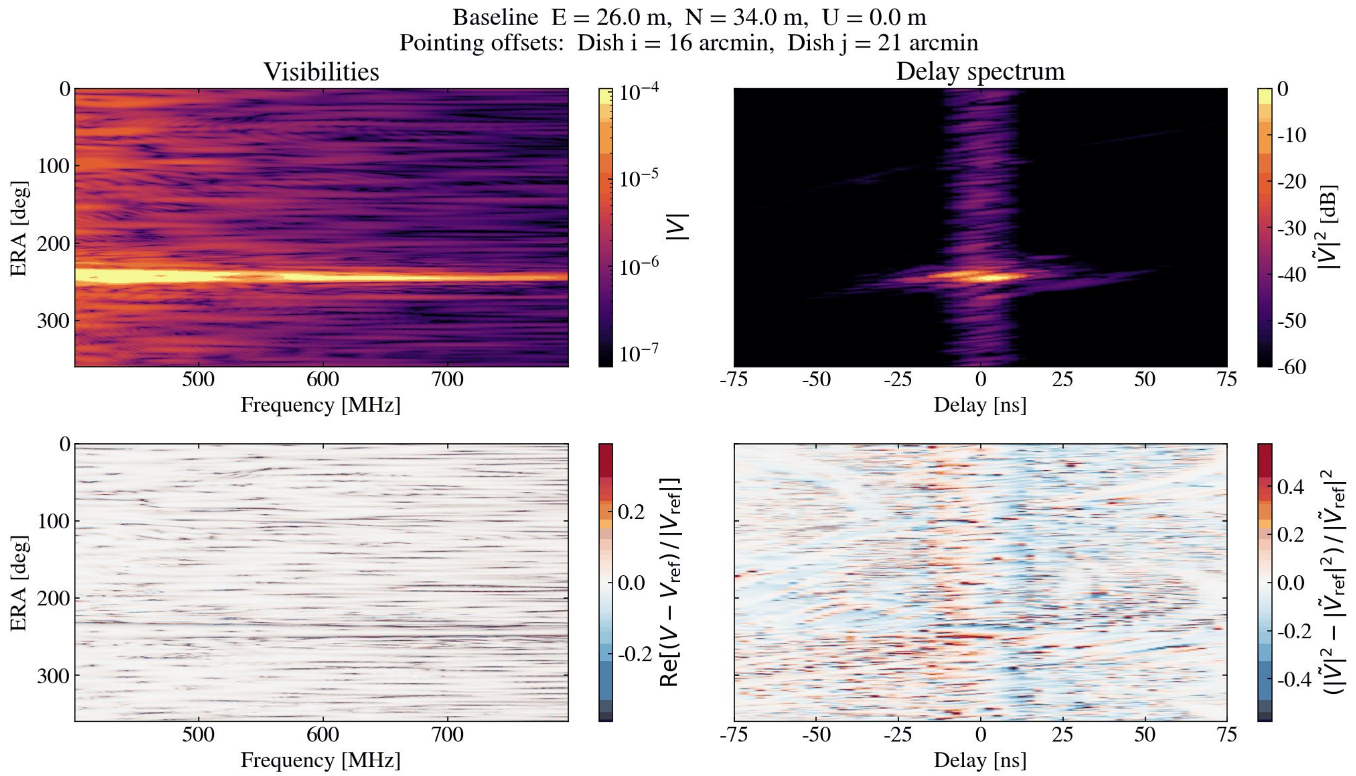

# Scaling - Cached vs Direct Visibility Synthesis

Example timings for 128 channel (400-800 MHz) fully sampled sidereal days on 128 core node, power beam lmax: 200

$$\mathcal{V}^{ij\nu}_{m} = \sum_{\ell_p,m_p} P^{i-j,\nu}_{l_p m_p m} p^{ij,\nu}_{\ell_p m_p}$$

- Visibility generation trivial post cache generation
- Cache generation pays off after a moderate number of beam evaluations (baseline dependent)

| Baseline | Direct per beam pair* [min] | Cache Generation [min] | N beam pair direct per cache gen. | N beam pairs HIRAX 128 elem. (full pol.) |
|---|---|---|---|---|
| E: 6.5 m, N: 8.5 m (~30$\lambda_{min}$) | 3.2 | 21 | 7 | 484 |
| E: 13 m, N: 17 m (~60$\lambda_{min}$) | 4.0 | 63 | 16 | 400 |
| E: 26 m, N: 34 m (~120$\lambda_{min}$) | 6.9 | 220 | 33 | 256 |

*Comparison not perfect. Direct could be done faster and IO overheads can have an affect.

# Conclusions

Future and ongoing work:

- Generating a library of caches for HIRAX baselines and standard sky models
- Utilising this to produce visibilities from simulated HIRAX beam responses with systematics perturbations and non-redundancies
- Incorporating into (quasi-)redundant calibration pipelines and e.g. CorrCal model building
- Propagating current understanding of expected HIRAX systematics into visibility-space statistics
- Developing simulations-informed, systematics-hardened foreground mitigation for cosmological analyses.

**Thank you!**